Measuring the Mass of the Earth: The Ultimate Moonbounce Experiment

By H. Paul Shuch, N6TX
Professor of Electronics
Pennsylvania College of Technology
One College Avenue
Williamsport, PA 17701

Abstract

Radio amateurs have been successfully bouncing VHF, UHF and microwave signals off the surface of the moon, and receiving their echoes, for nearly forty years. EME (Earth-moon-Earth) activities have enhanced the teaching of disciplines as diverse as electronics, astronomy, and physics, with several generations of students having now used measured echoes to determine the distance to, and orbital parameters of, the moon. Penn College students have recently had an opportunity to apply these measurements on a truly grand scale, using EME signals to measure the mass of the Earth. Their results differ from the currently accepted figure by about one percent.

STATEMENT OF THE PROBLEM

Give me a lever long enough, and I'll move the world. Give me a scale big enough, and I'll weigh it. This is essentially the problem posed to Electronics students at the Pennsylvania College of Technology in the Spring of 1991. Their innovative solution to weighing their home planet is worthy not only of the best scientific minds, but of the spirit of innovation with which the Amateur Radio Service has always prided itself.

The measurement procedure which they derived is based upon a Newtonian solution to the two-body orbital problem. The forces which hold a satellite in orbit around its primary are gravity (a force pulling in) and inertia (a force pulling out). To achieve a stable orbit, these two forces must of course be in equilibrium. Gravity varies directly with the mass of both objects; inertia, directly with the mass of only the satellite. Thus in setting the forces of gravity and inertia equal, the mass of the satellite cancels, leaving an expression for the orbit of the satellite which involves only the mass of the primary (plus a constant which Newton threw in for dimensional consistency).

The orbital characteristics of our natural satellite, the moon, are determined through a combination of visual observation and radio ranging. From them, we determine the velocity of the moon as it orbits the Earth. Applying Newton's Laws, we can then calculate the planetary mass required to produce the observed orbital velocity.

ESTABLISHING A MATHEMATICAL BASIS

As has been noted, a stable orbit requires that the forces of gravity and inertia be equal. Newton's famous Inverse Square Law shows the force of the gravitational attraction between any two bodies to equal a fudge factor (the Universal Gravitational Constant), times the product of their masses, divided by the square of the distance between their centers of mass. Mathematically,

$$F = \frac{GMm}{r^2} \qquad \text{(Eq 1)}$$

where M is the mass of the primary (planet),
m is the mass of the secondary (satellite),
r is the distance between them (the orbital radius),

and $G = \dfrac{6.673 \times 10^{-11}\ \text{Nt m}^2}{\text{kg}^2}$

Newton's Universal Gravitational Constant.

We now consider the force of inertia pulling a satellite out, which (again according to Newton) equals:

$$F = mA \qquad \text{[Eq 2]}$$

where m represents the mass of the satellite,
and A is its acceleration, which in a circular orbit is found from:

$$A = \frac{v^2}{r} \qquad \text{(Eq 3)}$$

with v representing the velocity of the satellite,
and r the orbital radius, as defined above.

Combining Eqs [2] and [3] gives us:

$$F = \frac{m\,v^2}{r} \qquad \text{(Eq 4)}$$

from which we can determine the inertial force acting on the moon, if we know its orbital velocity. It is this latter item which we will derive from our EME experiment.

Since the moon appears to be in a stable orbit, we set Equations [1] and [4] equal to each other, and then simplify:

$$\frac{GMm}{r^2} = \frac{m\,v^2}{r} \qquad \text{(Eq 5)}$$

$$\frac{GM}{r} = v^2$$

$$M = \frac{v^2\,r}{G} \qquad \text{(Eq 6)}$$

and all that remains is to measure orbital radius, calculate velocity, and solve for the mass of the Earth.

DETERMINING THE ORBITAL PERIOD OF THE MOON

This is perhaps the easiest part of the experiment. The orbital period of the moon is readily measured by eye, given the proper precision laboratory apparatus: a calendar. To the untrained observer, the elapsed time between two successive full (or new) moons appears to be on the order of about twenty-eight days. To the skilled scientist, on the other hand, the measurement comes out more like four weeks.

How long does it take the moon to orbit the Earth? Where do you think the word "month" comes from? (OK, by rights it should be "moonth," but my students will tell you I seldom take off for spelling). If we assume an orbital period of 28 days, and apply a bit of dimensional analysis, we get:

$$P = (28 \text{ days}) \times (24 \text{ hours/day}) \times (60 \text{ min/h}) \times (60 \text{ s/min});$$

$$P = 2{,}419{,}200 \text{ seconds, or:}$$

$$P = 2.42 \times 10^6 \text{ s}$$

We will employ this figure shortly, in computing the moon's orbital velocity.

FINDING THE ORBITAL RADIUS

Remember that the radius of an orbit is measured as the distance between the centers of mass of the two bodies. If we assume the centers of mass of the Earth and the moon to each be at the objects' physical center, then the orbital radius becomes the sum of the respective radii, plus the shortest distance between the surfaces of the two bodies. The radii we can calculate; it is the physical separation which we measure next.

(A) Distance from Earth to Moon

Those hundreds of radio amateurs who have experienced the thrill of hearing their own lunar echoes know that the elapsed time between transmission and reception is on the order of two and a half seconds. What we require here is a more precise estimate of echo time, measured when the moon is directly above the observer.

Unfortunately, the moon's orbit is aligned more or less with the Earth's equatorial plane, so the moon never passes directly over Penn College. There is about a twenty degree tilt to the lunar orbital plane, so when the moon is at maximum northern declination, and as its hour angle approaches local longitude, it appears *nearly* overhead from much of North America. Still, to minimize measurement error, the distance should properly be measured with the moon at zenith, and with a declination equal to local latitude. The present experiment utilized audio tapes of EME echoes made from a more southern locale, with the moon directly overhead, to determine minimum echo time.

You can get a fair estimate of round-trip propagation time by starting a stopwatch when you key your transmitter, and stopping it when you hear the echo. If you're working from audio tapes, start your clock on the transmitter's sidetone, and stop it on the received echo audio tone. For greater accuracy, my students chose to apply the tones to an oscilloscope, and use its calibrated timebase to measure the echo time, which came to 2.55 seconds.

We know radio waves to be propagating at the speed of light, $c = 2.998 \times 10^8$ meters per second. Multiplying speed by time yields distance: 7.645×10^8 meters round trip, 3.83×10^8 meters one-way, or about 238,000 miles.

(B) Radius of the Earth

Here we turn to the ancients for guidance. One of the earliest accurate measurements of the Earth's size involved observing that, on the day of the Equinox at local noon in Alexandria (which is near the equator), a stick placed vertically in the ground cast no shadow. However a stick similarly positioned in Athens (some known distance to the north) cast at the same time a shadow of appreciable length. Applying a bit of trigonometry, the Greeks computed the size of the Earth rather precisely.

We could readily duplicate their experiment. Or accept its result on faith. Either way, we have a dimension for the Earth's radius which falls rather close to the currently accepted "exact" value of 6.37×10^6 meters, which we will employ in the computations which follow.

(C) Radius of the Moon

Now I'm not going to suggest that we measure the lengths of shadows cast by sticks on the lunar surface (although one of my students did suggest that would make for an interesting field trip). Rather, we can observe the moon from Earth, and estimate its radius through trigonometry.

Any number of simple optical instruments tell us that, as viewed from Earth, the moon subtends an angle of about a half a degree. Given a quarter-degree "half angle" from the moon's center to its limb, and the length of the adjacent side (computed from our EME echoes), the tangent function gives us the opposite side of a right triangle (that is, the lunar radius) as:

$$\tan(1/4^\circ) \times d = r_m$$
$$\tan(1/4^\circ) \times 3.83 \times 10^8 \text{ meters} = r_m$$
$$r_m = 1.67 \times 10^6 \text{ meters}$$

or about a quarter of the radius of the Earth.

(D) It All Adds Up

The moon's orbital radius is now found simply by adding the one-way EME path length, the radius of the Earth, and the radius of the moon:

$$r = 383 \times 10^6 + 6.37 \times 10^6 + 1.67 \times 10^6 \text{ meters}$$

$$r = 391 \times 10^6 \text{ meters, or about 243,000 miles.}$$

CALCULATING THE LUNAR ORBITAL VELOCITY

Since distance always equals velocity times time, we can find the orbital velocity of the moon by dividing the distance it travels in one orbit, by the time it takes to complete one orbit. The distance traveled in an orbit is of course orbital circumference, which is found by multiplying orbital radius by 2 pi (the number of radians in a circle). The corresponding time is simply the orbital period (one "moonth"). Computing velocity:

$$v = d / t$$

$$v = C / P$$

$$v = \frac{(2 \times \pi \times 391 \times 10^6 \text{ meters})}{(2.42 \times 10^6 \text{ s})}$$

$$v = 1015 \text{ meters per second}$$

SOLVING FOR THE EARTH'S MASS

Stick around, class, we're almost done. Inserting the orbital radius and velocity into Eq 6, we get:

$$M = \frac{v^2 r}{G} \qquad \text{(Eq 6)}$$

$$M = \frac{(1015\ \text{m/s})^2 \times (391 \times 10^6\ \text{m})}{(6.673 \times 10^{-11}\ \text{Nt m}^2/\text{kg}^2)}$$

$M = 6.037 \times 10^{24}$ kg, which is a heavy planet indeed!

EVALUATING OUR RESULT

Any astronomy textbook will reveal multiple measurement errors for each of the intermediate values which we used to estimate the mass of the Earth. For example, the actual period of the lunar orbit is about two percent less than we estimated through our rather crude calendar technique. The lunar radius is in fact nearly four percent greater than we estimated through visual observation and simple trigonometry. And the radius of the moon's orbit turns out to be about a percent and a half less than we calculated from our echo measurements.

With all the above measurement errors, one might expect our experimental result (the computed mass of the Earth) to be significantly in error. It turns out otherwise. As is often the case in computations based upon multiple independent measurements of unrelated phenomena, our various errors tend to cancel out, yielding a result which would do Pythagoras proud.

Well, how close is our final result? The published value for the mass of the Earth is: 5.975×10^{24} kg, so we see we're rather close. To determine the percentage of error between observed and theoretical values, subtract the theoretical from the observed, divide by the observed, and then (in order to convert ratio to percentage), multiply by a hundred. In our case:

$$\%\ \text{Error} = 100 \times \left[(6.037 \times 10^{24}) - \frac{(5.975 \times 10^{24})}{(5.975 \times 10^{24})} \right]$$

$\%\ \text{Error} = 1.03\%$

Isn't it amazing how accurate we can be, using direct observation and strictly amateur techniques?

CONCLUSIONS

The EME challenge has been successfully met by hundreds, perhaps thousands of radio amateurs eager to expand their communications horizons. It also provides us with a mechanism for training the next generation of engineers and technicians. More important, it serves as a source of motivation and inspiration for those whom society will ask to develop technology to cope with the challenges of the future. Weighing the earth may appear frivolous, but certainly not mundane. It demonstrates to students the diverse applications of the radio art, and provides them with a verifiable problem of truly cosmic proportions, against which to weigh themselves. Once they've held the Earth in their hands, how can they not be moved?

Our students are our future. Of course, we on the campuses will help them to acquire knowledge, and to master skills. But we must not stop there. EME (the Earth Mass Experiment) and similar experiences can provide them with something far more precious: motivation. Give them a lever long enough, and they will move the world!

EME REFERENCES

Tilton, E., "Lunar DX on 144 Mc," *QST* Mar 1953, pp 11-12, 116.

Orr, W. and Harris, S., "Project Moon Bounce," *QST*, Sept 1960, pp 62-66, 158.

Turin, D. and Katz, A., "Earth-Moon-Earth (EME) Communications," *ARRL UHF/Microwave Experimenter's Manual*, 1990, ARRL, Newington, CT, Chapter 10.

ACKNOWLEDGEMENT

This paper was originally presented at the *25th Conference of the Central States VHF Society*, July 27, 1991, Cedar Rapids, IA.

ABOUT THE AUTHOR

H. Paul Shuch began his microwave experiments in the 1960s, using APX-6 transponders at 1215 MHz. He heard his first EME echo in 1973 and has not been the same since. He has operated on all 19 ham bands between 1.8 MHz and 10 GHz. Paul holds a PhD in Engineering from the University of California, Berkeley, has taught for nearly twenty years, has published half a hundred technical articles in engineering and scholarly journals, and was a coauthor of the *ARRL UHF/Microwave Experimenter's Manual*.